The boundary layer functions are sought in the form (2.5). The procedure for constructing them is analogous to that described above with the sole difference that the functions $v_{n,0}(\eta, \theta)$ will satisfy boundary conditions of the Neumann type for $\eta = 0$, which enables $g_n'(0)$, $g_n'(x_0)$ to be determined from the condition for the damping of the boundary layer functions to be exponential /3/.

The asymptotic form of problem B_ε differs substantially from the asymptotic form of problem A_ε in that the series expansion in powers of ε for $v(x, \theta)$ must start with the power -2, and this is related, in turn, to the fact that the coefficient of friction is assumed to be non-zero. For $\rho = 0$ the series expansion starts with the zeroth power of ε.

The system of equations (2.2) to determine the functions u_n, v_n is hyperbolic with two double families of characteristics $x \equiv \text{const}$ and $\theta \equiv \text{const}$, which indeed results in the appearance of the average with respect to the angular coordinate in the asymptotic form because of the requirement for the displacement to be unique. We note that the "radial" part of the functions $v_n(x, \theta)$ is extracted automatically in problem B_ε.

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THE PROBLEM OF THE CONTACT BETWEEN A LINEAR ELASTIC BODY AND ELASTIC AND RIGID BODIES (A VARIATIONAL APPROACH)*

A.M. KHLUDNEV

The problem of the contact between a linear elastic body and a rigid body is formulated as a one-sided problem. The solution is determined from the variational inequality, equivalent to the problem of minimizing the energy functional in a set of allowable displacements. The regularity of the solution is established down to internal points of the contact boundary. A measure is constructed in the subsets of the contact boundary that enables the effect of a stamp on an elastic body to be characterized. The absolute continuity of this measure is proved at the internal point. The problem of the contact of two elastic bodies is examined in a similar formulation. The regularity of the solution is established and the nature of the effect of one body on the other is clarified.

1. Contact between an elastic and a rigid body.

Formulation of the problem. Let an elastic body in the natural state occupy a domain $\Omega \subset R^3$ with boundary Γ of class C^∞ represented in the form of the union of three parts: $\Gamma = \Gamma_\omega \cup \Gamma_\sigma \cup \Gamma_c$. The condition $\omega = 0$ is given on Γ_ω, where ω is the displacement vector. The vector force $\sigma_{ij} n_j = g_i$ is given on Γ_σ, where $n = (n_1, n_2, n_3)$ is the external normal to the boundary, σ_{ij} is the stress tensor, g_i are given surface forces, $i, j = 1,2,3$, and summation here and below is over repeated subscripts. It is assumed that the points Γ_c of the elastic body can interact with the rigid body for which the equation of the surface has the form $\Phi(x) = 0$, where the inequality $\Phi(x) \leqslant 0$ is satisfied for points of the rigid body. In the linear approximation the condition on the displacement vector has the form /1/

$$\omega(x) \nabla \Phi(x) \geqslant -\Phi(x), \quad x \in \Gamma_c \tag{1.1}$$

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Let $H_\omega^1(\Omega)$ be a Sobolev space of vector functions having first generalized, square-summable derivatives in Ω that vanish on Γ_ω ($\operatorname{mes}\Gamma_\omega > 0$). Let K denote the set of functions from this space that satisfy the inequality (1.1) almost everywhere on Γ_c (in the plane measur sense) by considering Γ_c to be a simply-connected domain Γ with a smooth boundary $\partial\Gamma_c$ of class C^1.

The solution of the problem of minimizing the energy functional

$$\Pi(\omega) = \frac{1}{2}\int\limits_{\Omega} \sigma_{ij}(\omega)\,\varepsilon_{ij}(\omega)\,dx - \int\limits_{\Omega} f_i u_i\,dx - \int\limits_{\Gamma_\sigma} g_i u_i\,d\Gamma, \quad \omega = (u_1, u_2, u_3) \tag{1.2}$$

in the set K satisfies the inequality

$$\omega \in K\colon (d\Pi(\omega), \chi - \omega) \geqslant 0 \quad \forall \chi \in K \tag{1.3}$$

$d\Pi(\omega)$ is the gradient of $\Pi(\omega)$ on $H_\omega^1(\Omega)$. If the shape of the stamp here agrees with Γ_c, then (1.3) is the classical Signorini problem /2/. For simplicity, we will consider the case of an elastic body which obeys Hooke's law.

The smoothness of the solution. It can be proved that if $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_\sigma)$, the solution of (1.3) exists and is unique.

In investigating the qualitative properties of the solution, the proof of its regularity in the neighbourhood of the points of Γ_c is essential. We assume everywhere that the shape of the stamp does not differ very much from Γ_c. The exact meaning of this condition is clarified below.

Theorem 1. Let the function Φ belong to the class C^3. Then the solution $\omega \in K$ possesses two generalized derivatives in the neighbourhood of the points $x \in \Gamma_c \setminus \partial\Gamma_c$.

Proof. Let $x_0 \in \Gamma_c \setminus \partial\Gamma_c$. We select a system of coordinates with origin at the point x_0 such that the plane x_1x_2 is tangent to Γ_c at x_0, while the axis x_3 coincides with the interio normal at this point. Let

$$x_3 = \beta(x_1, x_2), \quad \beta_{x_1}(0,0) = \beta_{x_2}(0,0) = 0 \tag{1.4}$$

be the equation of Γ_c in the neighbourhood of x_0 and

$$x_3 = \alpha(x_1, x_2) \tag{1.5}$$

the equation of the shape of the stamp. Such a representation of the boundary is possible because of its smoothness. The meaning of the assumption made above is that the shape of the stamp does not differ very much from Γ_c, which again consists of the possibility of the representation (1.5). Inequality (1.1) has the following form in the neighbourhood of the point x_0 ($u = u_1$, $v = u_2$, $w = u_3$):

$$\begin{aligned} &-\alpha_{x_1}(x_1, x_2)\,u(x_1, x_2, \beta(x_1, x_2)) - \alpha_{x_2}(x_1, x_2)\,v(x_1, x_2, \beta(x_1, x_2)) + \\ &w(x_1, x_2, \beta(x_1, x_2)) \geqslant \alpha(x_1, x_2) - \beta(x_1, x_2), \quad (x_1, x_2) \in D \end{aligned} \tag{1.6}$$

Here D is the projection of the boundary (1.4) on the plane x_1x_2.

Let us make a change of variables with a unit Jacobian

$$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3 - \beta(x_1, x_2)$$

The following formulas will be valid here for an arbitrary smooth function $h(x) \equiv \bar{h}(y)$:

$$h_{x_1} = \bar{h}_{y_1} - \bar{h}_{y_3}\beta_{x_1},\ h_{x_2} = \bar{h}_{y_2} - \bar{h}_{y_3}\beta_{x_2},\ h_{x_3} = \bar{h}_{y_3}$$

The transformation mentioned will result in the fact that a fairly small domain $\Omega_0 \subset \Omega$ having the surface $x_3 = \beta(x_1, x_2)$ as part of its boundary will be mapped homeomorphically onto a domain in the space of variables y with a boundary containing pieces of the plane $y_3 = 0$. Let $y(\Omega_0)$ be this mapping. We select the domain in the form of hemispheres

$$G_i = \{\,|y| < i\delta,\ y_3 > 0\},\ i = 1, 2, 3$$

where we consider δ to be such a small positive number that $G_3 \subset y(\Omega_0)$. Furthermore, let the function $\varphi(y) \in C^\infty$ possess the properties $\varphi \equiv 1$ in $\bar{G}_1$, $\varphi \geqslant 0$, $|\varphi| \leqslant 1$, where $\varphi \equiv 0$ outside $|y| \geqslant {}^3/_2\delta$, and we assume that the second derivatives of the function $\alpha - \beta$ with respect to y_1, y_2 are non-negative in G_3. The assumption made about the sign of the second derivatives of the function $\alpha - \beta$ is not essential and can be dropped. We also introduce the following notation

$$d_{i\tau}\,h(y) = [h(y + \tau e_i) - h(y)]\tau^{-1},\ \Delta_{i\tau}h(y) = -d_{-i\tau}d_{i\tau}h(y), \qquad i = 1, 2$$

where e_i are the unit vectors, and $0 < \tau < \delta$. It can be shown that if the number λ is selected in the range $0 < \lambda < {}^1/_2\tau^2$, the vector $\bar{\chi}_\lambda = (\bar{u}_\lambda, \bar{v}_\lambda, \bar{w}_\lambda)$ with the components

$$\bar{u}_\lambda = \bar{u} + \lambda\varphi^2\Delta_{i\tau}\bar{u},\ \bar{v}_\lambda = \bar{v} + \lambda\varphi^2\Delta_{i\tau}\bar{v}$$

$$\bar{w}_\lambda = \bar{w} + \lambda\varphi^2\Delta_{i\tau}[\bar{w} - \bar{u}\alpha_{y_1} - \bar{v}\alpha_{y_2}] + \lambda\varphi^2\alpha_{y_1}\Delta_{i\tau}\bar{u} + \lambda\varphi^2\alpha_{y_2}\Delta_{i\tau}\bar{v}$$

will satisfy inequality (1.6) written in the variables y.

The carrier $\chi_\lambda - \omega$ lies in the set $\Omega_0 \cup \{x_3 = \beta(x_1, x_2)\}$. For any sufficiently regular function p,h such that the carrier p lies in the set mentioned, the following equation holds:

$$\int_\Omega h_{x_i} p_{x_j}\, dx = \int_{y(\Omega_0)} (\bar h_{y_i} - \bar h_{y_3}\beta_{x_i})(\bar p_{y_j} - \bar p_{y_3}\beta_{x_j})\, dy, \quad \beta_{x_3} = 0$$

We now substitute the function $\chi_\lambda = (u_\lambda, v_\lambda, w_\lambda)$ into (1.3) and divide by λ. Because of the preceding relationship, we will have two types of components in the inequality obtained: those containing and those not containing β_{x_1}, β_{x_2}. The following assertion holds for the components not containing β_{x_i}. The difference between the integrals (there is no summation with respect to i here and henceforth)

$$\int_{y(\Omega_0)} \bar u_{y_k} (\varphi^2 \Delta_{i\tau} \bar v)_{y_l}\, dy, \quad - \int_{y(\Omega_0)} (d_{i\tau}\varphi \bar u)_{y_k} (d_{i\tau}\varphi \bar v)_{y_l}\, dy$$

can have as upper bound the quantity included in the second braces on the right side of inequality (1.7) (see below) with a constant dependent only on $y(\Omega_0)$ and φ. Here each of the functions can independently take the values $\bar u, \bar v, \bar w$.

We extract the highest from the expressions containing β_{x_i}. They can be estimated as before and the maximum of the quantities $\beta_{x_i}, \beta_{x_i}^2$ in absolute value can be taken outside the integral sign. The other quantities can be estimated more simply. We consequently conclude that the following inequality holds:

$$\int_{y(\Omega_0)} \sigma_{kl}(d_{i\tau}(\varphi\bar\omega))\, \varepsilon_{kl}(d_{i\tau}(\varphi\bar\omega))\, dy \leqslant \tag{1.7}$$

$$c_1 \max_{\bar G_3} \{|\beta_{x_1}| + |\beta_{x_2}| + \beta_{x_1}^2 + \beta_{x_2}^2\} \, \| d_{i\tau}(\varphi\bar\omega)\|_1^2 + c_2 \{ \|\bar\omega\|_1^2 + \|\bar\omega\|_1 \| d_{i\tau}(\varphi\bar\omega)\|_1 + \|\bar f\|_0^2\}$$

where c_2 depends on the domain $y(\Omega_0)$, the Lamé constants, and the function φ, α, β; while c_1 depends only on the Lamé constants. All the norms written down are referred to the domain $y(\Omega_0)$.

Furthermore, we use the Korn inequality

$$\| d_{i\tau}(\varphi\bar\omega)\|_1^2 \leqslant c_0 \int_{y(\Omega_0)} \sigma_{kl}(d_{i\tau}(\varphi\bar\omega))\, \varepsilon_{kl}(d_{i\tau}(\varphi\bar\omega))\, dy \tag{1.8}$$

Here the constant c_0 is independent of $\bar\omega$ and φ. Selecting the quantity δ characterizing the domain G_i to be sufficiently small, we can assume that

$$\max_{\bar G_3} \{|\beta_{x_1}| + |\beta_{x_2}| + \beta_{x_1}^2 + \beta_{x_2}^2\} < (c_0 c_1)^{-1}$$

where c_0 and c_1 are the constants from (1.7) and (1.8). This selection is possible because of conditions (1.4). Therefore, we obtain from (1.7)

$$\| d_{i\tau}(\varphi\bar\omega)\|_1 \leqslant c$$

with a constant c independent of τ. Therefore, the second derivatives of the function $\bar\omega$, with the exception of $\bar\omega_{y_3 y_3}$, belong to $L^2(G_1)$. However, it is seen that the equations

$$\bar\omega_{y_3 y_3} = F$$

are satisfied in the neighbourhood of $y_3 = 0$, where $F \in L^2(G_1)$. Thus the second derivatives of the solution with respect to y_3 also belong to $L^2(G_1)$. The theorem is proved.

Construction of the measure. The assertion about the existence of a measure characterizing the action of the stamp on an elastic body will be proved below. The case when Γ_c has a common boundary with Γ_σ and when it has one with Γ_ω should be distinguished here. First let the points of $\partial\Gamma_c$ possess the following property: for any $x_0 \in \partial\Gamma_c$ a neighbourhood $d(x_0)$ exists such that $d(x_0) \cap \Gamma \subset \Gamma_c \cup \Gamma_\sigma$.

Theorem 2. A measure μ can be given in a σ-algebra of Borel subsets of the boundary Γ_c such that for all $\chi \in H_\omega^1(\Omega) \cap C(\Omega \cup \Gamma_c)$ the following representation holds:

$$(d\Pi(\omega), \chi) = \int_{\Gamma_c} \frac{\chi\nabla\Phi}{|\nabla\Phi|}\, d\mu \tag{1.9}$$

Proof. We first note that if the vector $\chi \in H_\omega^1(\Omega)$ is such that $\chi\nabla\Phi \geqslant 0$ on Γ_c, then $(d\Pi(\omega), \chi) \geqslant 0$. In fact, $\omega + \varepsilon\chi \in K$, $\varepsilon > 0$, hence by substituting $\omega + \varepsilon\chi$ as a trial vector in (1.3), we obtain the required inequality. We now define a linear manifold of functions given on Γ_c

$$V = \{\chi_*\}, \quad \chi_* = \frac{\chi\nabla\Phi}{|\nabla\Phi|}, \quad \chi \in H_\omega^1(\Omega) \cap C(\Omega \cup \Gamma_c) \tag{1.10}$$

The linear functional

$$\psi(\chi_*) = (d\Pi(\omega), \chi)$$

can be defined on V.

The value of ψ is determined uniquely by this formula. Indeed, if $\chi_*^1 = \chi_*^2$ on Γ_c, then according to the preceding $(d\Pi(\omega), \chi^1 - \chi^2) \geqslant 0$. Since the reverse inequality is also true, we have $\psi(\chi_*^1) = \psi(\chi_*^2)$.

It can be shown that the manifold V contains all functions from $C^1(\Gamma_c)$, and hence, its closure in the norm $\|\cdot\|_{C(\Gamma_c)} = \max\limits_{\Gamma_c} |\cdot|$ agrees with the space $C(\Gamma_c)$. Moreover, a vector $h \in H_\omega^1(\Omega) \cap C(\Omega \cup \Gamma_c)$ exists such that

$$h\nabla\Phi/|\nabla\Phi| \geqslant \delta > 0, \ \delta = \text{const}$$

Hence, for any function $\chi_* \in V$

$$|\chi_*| \leqslant \|\chi_*\|_{C(\Gamma_c)} h\nabla\Phi/(\delta|\nabla\Phi|)$$

It therefore follows from the fact that the functional ψ is positive that

$$|\psi(\chi_*)| \leqslant c\|\chi_*\|_{C(\Gamma_c)}, \ c = \delta^{-1}(d\Pi(\omega), h)$$

Therefore, we conclude that ψ is a linear continuous functional on $C(\Gamma_c)$. Since it is moreover positive, a measure μ exists such that

$$\psi(\chi_*) = \int_{\Gamma_c} \chi_* \, d\mu \quad \forall\chi_* \in C(\Gamma_c)$$

For a function χ_* constructed according to the vector $\chi \in H_\omega^1(\Omega) \cap C(\Omega \cup \Gamma_c)$ by using (1.10), this formula yields the representation (1.9). The proof of the theorem is completed.

Absolute continuity of the measure in $\Gamma_c \setminus \partial\Gamma_c$. Let $x_0 \in \Gamma_c \setminus \partial\Gamma_c$. A sufficiently small neighbourhood $d(x_0)$ of the point x_0 exists that does not contain the points $\partial\Gamma_c$. Let the vector $\chi = (\chi_1, \chi_2, \chi_3) \in H_\omega^1(\Omega) \cap C(\Omega \cup \Gamma_c)$ be such that $\operatorname{supp}\chi \in d(x_0) \cap \bar{\Omega}$. Because of Theorem 2.

$$(d\Pi(\omega), \chi) = \int_{\Gamma_c} \frac{\chi\nabla\Phi}{|\nabla\Phi|} \, d\mu \tag{1.11}$$

Using the result about the regularity of the solution in the neighbourhood of the point x_0, we conclude that the left side of this relationship equals

$$\int_{\Omega} \left\{\sigma_{lj} \frac{\partial\chi_l}{\partial x_j} - f_l\chi_l\right\} dx = \int_{\Gamma_c} \sigma_{lj} n_j \chi_l \, d\Gamma \tag{1.12}$$

Furthermore, we recall the definition. A measure γ is called singular with respect to the Lebesgue measure if it is concentrated in a set of zero Lebesgue measure. For an arbitrary measure given on a σ-algebra of Borel sets of Γ_c (and particularly for μ), there exists a unique decomposition

$$\mu(B) = \gamma(B) + \int_B q(x) \, dx$$

Here $q(x)$ is a function summable in the Lebesgue measure (the density of the measure μ), and $B \subset \Gamma_c$ is an arbitrary Borel set.

We will show that $\gamma \equiv 0$ in $\Gamma_c \setminus \partial\Gamma_c$ follows from (1.11) and (1.12). Hence, a deduction can be made, in particular, about the impossibility of concentrated actions of the stamp on an elastic body at the point $\Gamma_c \setminus \partial\Gamma_c$. By our assumption, $|\nabla\Phi| \neq 0$ on Γ_c. Let a coordinate system be selected such that $\Phi_{x_i}(x) \neq 0$ for $x \in d(x_0) \cap \Gamma_c$, $i = 1, 2, 3$. Considering, in turn, that the non-zero component of χ is only χ_i and equating the right sides of (1.11) and (1.12), we obtain (no summation over i)

$$\int_{\Gamma_c} \chi_i \frac{\Phi_{x_i}}{|\nabla\Phi|} \, d\mu = \int_{\Gamma_c} \sigma_{ij} n_j \chi_i \, d\Gamma$$

Hence, it follows that the singular component of the measure μ is equal to zero in $d(x_0) \cap \Gamma_c$, where the measure density equals

$$q = \sigma_{ij} n_j |\nabla\Phi|/\Phi_{x_i}, \quad i = 1, 2, 3$$

Let σ denote a vector with components $\sigma_{ij} n_j$, $i = 1, 2, 3$. Then the density can be written in the form

$$q \equiv q\frac{\Phi_{x_1}^2}{|\nabla\Phi|^2} + q\frac{\Phi_{x_2}^2}{|\nabla\Phi|^2} + q\frac{\Phi_{x_3}^2}{|\nabla\Phi|^2} = \frac{\sigma\nabla\Phi}{|\nabla\Phi|}$$

Let us now examine the case when Γ_c has a common boundary with Γ_ω. In other words, for an arbitrary point $x_0 \in \partial\Gamma_c$ let a neighbourhood $d(x_0)$ exist for which $d(x_0) \cap \Gamma \subset \Gamma_c \cup \Gamma_\omega$.

Theorem 3. A measure μ can be given on a σ-algebra of Borel subsets $\Gamma_c \setminus \partial\Gamma_c$ such that for all $\chi \in H_\omega^1(\Omega) \cap C_0(\Gamma_c)$ the representation (1.10) holds.

That $C_0(\Gamma_c)$ belongs to the space of finite and continuous functions on Γ_c should here be understood to mean that the traces of the components of the vector χ in Γ_c belong to the space mentioned.

We will make some remarks about the proof of the theorem. In this case a linear manifold V consisting of the functions χ_* of the form

$$\chi_* = \chi\nabla\Phi / |\nabla\Phi|, \ \chi \in H_\omega^1(\Omega) \cap C_0(\Gamma_c)$$

will contain all the functions from $C_0^1(\Gamma_c)$. Therefore, a linear functional ψ on V defined by the formula $\psi(\chi_*) = (d\Pi(\omega), \chi)$ can be defined in continuity for all functions from $C_0(\Gamma_c)$. Here (1.10) will follow from the well-known representation of a linear continuous functional in the space of finite continuous functions.

In this case the density of the measure μ also equals $\sigma\nabla\Phi / |\nabla\Phi|$.

2. The contact of two elaatic bodies. *Formulation of the problem.* Let two elastic bodies in the natural state, occupying the domains Ω and Ω' from R^3 with boundaries Γ, Γ', have common pieces of the boundary Γ_c. If ω is the displacement vector of points of the first body, and ω' of the second, then the fundamental inequality relating these vectors on Γ_c has the form /3/

$$\omega n - \omega' n \leqslant 0 \quad \text{on} \quad \Gamma_c \tag{2.1}$$

($n = (n_1, n_2, n_3)$ is the vector of the outer normal to Γ). The prime denotes quantities referring to the second body.

The fundamental result is the proof of the regularity of the solution up to Γ_c and the investigation of the qualitative properties of the solution. Exactly as in Sect.1, the solution (ω, ω') will be determined from the variational inequality. We first determine the boundary conditions. We consider that $\Gamma = \Gamma_c \cup \Gamma_\omega \cup \Gamma_\sigma$, $\Gamma' = \Gamma_c \cup \Gamma_{\omega'} \cup \Gamma_{\sigma'}$. The displacement vector is given on Γ_ω, and the vector of the forces on Γ_σ, i.e.,

$$\omega = 0 \quad \text{on} \quad \Gamma_\omega; \ \sigma_{ij} n_j = g_i \quad \text{on} \quad \Gamma_\sigma \tag{2.2}$$

Analogous conditions are given on $\Gamma_{\omega'}, \Gamma_{\sigma'}$. We assume that $\operatorname{mes}\Gamma_\omega > 0$, $\operatorname{mes}\Gamma_{\omega'} > 0$.

Let $H_{\omega'}^1(\Omega')$ have the same meaning as $H_\omega^1(\Omega)$. We use the notation $H = H_\omega^1(\Omega) \times H_{\omega'}^1(\Omega')$.

The energy functional of two bodies is represented in the form of the sum of appropriate functionals. For each of the bodies it has the form (1.2), where the Lamé parameters for each of the bodies is their own.

We assume that $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_\sigma)$ (similarly for the second body). Under the above-mentioned conditions, the solution of the problem of minimizing the energy functional $E(\omega, \omega') = \Pi(\omega) + \Pi'(\omega')$ in the closed convex set $K \subset H$, defined as the set of functions from H satisfying the inequality (2.1) almost everywhere in Γ_c, exists and is unique. This solution $\psi = (\omega, \omega')$ satisfies the inequality

$$\psi \in K: (dE(\psi), \chi - \psi) \geqslant 0 \ \forall \chi \in K \tag{2.3}$$

Here $dE(\psi)$ is the gradient of the functional $E(\psi)$ on H.

Problem (2.3) certainly allows a differential formulation, namely, the equilibrium equations

$$\partial\sigma_{ij}/\partial x_j = -f_i$$

are satisfied in the domains Ω and Ω'.

Conditions (2.2) are satisfied on Γ_ω and Γ_σ (on $\Gamma_{\omega'}$ and $\Gamma_{\sigma'}$, respectively) and, moreover, we have on Γ_c

$$\begin{cases} \omega n - \omega' n < 0 \\ \sigma_{ij}(\omega) n_j n_i = 0 \\ \sigma_\tau = 0 \end{cases} \quad \text{or} \quad \begin{cases} \omega n - \omega' n = 0 \\ \sigma_{ij}(\omega) n_j n_i \leqslant 0 \\ \sigma_\tau = 0 \end{cases}$$

$$\sigma_{ij}(\omega) n_j n_i = \sigma_{ij}(\omega') n_j n_i, \ (\sigma_\tau)_i = \sigma_{ij} n_j - (\sigma_{ij} n_j n_l) n_l$$

We later assume that $\Gamma, \Gamma' \subset C^\infty$, and we consider Γ_c to be a simply-connected domain on Γ with a smooth boundary of the class C^1.

The regularity of the solution. Theorem 4. For each point $x_0 \in \Gamma_c \setminus \partial\Gamma_c$ a neighbourhood $d(x_0) \subset \Omega \cup \Omega' \cup \Gamma_c$ exists such that $\omega \in H^2(d(x_0) \cap \Omega)$, $\omega' \in H^2(d(x_0) \cap \Omega')$.

Ideas analogous to those used in proving Theorem 1 are used to prove this theorem. The neighbourhood of the point $x_0 \in \Gamma_c \setminus \partial\Gamma_c$ is transformed into a circle in the space of the variables y by a special coordinate transformation sothat part of the boundary near the point x_0 transfers into points of the plane. Then a trial function is selected that satisfies (2.1) and enables multiplication of the equilibrium equations by the appropriate derivatives to be reproduced in difference form. This function is substituted into (2.3), which in the long run results in an inequality of the type (1.7). Therefore, the presence of square-summable second derivatives along the tangent directions and mixed derivatives, is established. From the conditions for the equilibrium equations to be valid near the boundary we also obtain the existence of square-summable second derivatives along the normal.

Construction of the measure and its properties. We will formulate a theorem on the existence of a measure characterizing the action of one body on another. Exactly as in the problem of the interaction between an elastic and a rigid body, the case of the distinct location of $\Gamma_c, \Gamma_\sigma, \Gamma_\omega, \Gamma_{\sigma'}, \Gamma_{\omega'}$, must be examined separately.

First, for each point $x_0 \in \partial\Gamma_c$ let a neighbourhood $d(x_0)$ exist that possesses the property that $d(x_0) \cap \Gamma \subset \Gamma_c \cup \Gamma_\sigma$ and $d(x_0) \cap \Gamma' \subset \Gamma_c \cup \Gamma_{\sigma'}$. Hence, the following theorem holds:

Theorem 5. A measure μ can be defined on a σ-algebra of Borel subsets Γ_c such that for arbitrary functions $\varphi = (\gamma, \gamma') \in H \cap C(\Gamma_c)$ the following representation holds ($\psi \in K$ is the solution of (2.3)):

$$(dE(\psi), \varphi) = -\int_{\Gamma_c} (\gamma n - \gamma' n)\, d\mu \tag{2.4}$$

The properties of the measure constructed in such a manner are determined by the smoothness of the solution. In particular, the presence of second derivatives for the solution near the contact boundary enables us to prove that the singular component of the measure μ equals zero at the points $\Gamma_c \setminus \partial\Gamma_c$. The reasoning is similar to that used at the end of Sect.1. The density of the measure μ turns out to equal $-\sigma_{ij}(\omega) n_j n_i$.

In conclusion, we consider the situation when a neighbourhood $d(x_0)$ exists for an arbitrary point $x_0 \in \partial\Gamma_c$ for which $d(x_0) \cap \Gamma \subset \Gamma_c \cup \Gamma_\omega$, $d(x_0) \cap \Gamma' \subset \Gamma_c \cup \Gamma_{\omega'}$.

Theorem 6. A measure μ can be defined on a σ-algebra of Borel subsets $\Gamma_c \setminus \partial\Gamma_c$ such that for any function $\varphi = (\gamma, \gamma') \in H \cap C_0(\Gamma_c)$ the representation (2.4) holds. The singular component of this measure is zero, and the density equals $-\sigma_{ij}(\omega) n_j n_i$, where $\mu(B) < +\infty$ for any compact $B \subset \Gamma_c \setminus \partial\Gamma_c$.

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ON THE FORMULATION AND INVESTIGATION OF A SPATIAL CONTACT PROBLEM FOR ELASTIC BODIES UNDER MIXED FRICTION CONDITIONS*

I.I. KUDISH

A spatial contact problem is formulated and investgated for rough elastic bodies which touch each other under mixed friction conditions: the elastic bodies are separated in one part of the contact domain by a layer of viscous incompressible liquid (lubricant), while in the other they are in direct contact (such conditions are characteristic for roller bearings, gear transmissions, etc.). The problem is reduced to a system of non-linear integro-differential and integral equations and inequalities in the contact domain, part of the external boundary, and a number of inner boundaries that are unknown in advance, but separate the lubricated and unlubricated zones. Special cases are problems of dry and completely

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